

Characterizing locally distinguishable orthogonal product states

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(Dated: February 1, 2008)

Bennett et al. [3] identified a set of orthogonal *product* states in the $3 \otimes 3$ Hilbert space such that reliably distinguishing those states requires non-local quantum operations. While more examples have been found for this counter-intuitive “nonlocality without entanglement” phenomenon, a complete and computationally verifiable characterization for all such sets of states remains unknown. In this Letter, we give such a characterization for the $3 \otimes 3$ space.

PACS numbers: 03.67.-a, 03.65.Ud, 03.67.Hk

A pure quantum state $|\phi\rangle_{AB}$ of a bipartite system AB is said to be entangled if it is not a product state, i.e., it cannot be represented as $|\alpha\rangle_A \otimes |\beta\rangle_B$, for some state $|\alpha\rangle_A$ and $|\beta\rangle_B$ of the system A and B , respectively. An entangled quantum state may generate measurement statistics that are inherently different from those generated by a classical process [1, 2]. This feature of entanglement is referred to as the nonlocality of quantum states. Dual to the notion of state nonlocality is the nonlocality of quantum operations. A natural definition of a local quantum operation on a multi-partite quantum system is that of *Local Operations and Classical Communication (LOCC)* protocols, in which each party may apply to his system arbitrary quantum operations, while the inter-partite communication must be classical. It follows from the definition that no LOCC protocol creates quantum entanglement. However, the reverse is false. This surprising fact was discovered by Bennett et al. [3] and was formulated as a problem of reliably distinguishing quantum states.

A set of state $\mathcal{E} = \{|\phi_i\rangle_{AB}\}_i$ is said to be *reliably distinguishable* by a quantum operation T if on each $|\phi_i\rangle_{AB}$, T outputs i with probability 1. The authors of [3] identified an orthonormal basis \mathcal{B}_9 for $\mathbb{C}^3 \times \mathbb{C}^3$, illustrated in Fig. 1, that cannot be reliably distinguished by LOCC. The important feature of the basis is that each base vector is a product state, thus the distinguishing operator cannot create entanglement.

The above property of nonlocal operations not necessarily creating entanglement is referred to as “nonlocality without entanglement”, and has been studied by many authors subsequently [3–15]. Formally, an *orthogonal product set (OPS)* is a set of bipartite states which are product states and are pairwise orthogonal. An OPS that forms a basis is also called an *orthogonal product basis (OPB)*. Much effort has been devoted to searching for additional LOCC-indistinguishable OPSs. Besides \mathcal{B}_9 , Ref. [3] actually showed that $\mathcal{B}_8 \stackrel{\text{def}}{=} \mathcal{B}_9 - \{|1\rangle|1\rangle\}$ is not LOCC-distinguishable, either. All other known LOCC-indistinguishable OPSs belong to the following

		B			
		0	1	2	
A	0	$ \phi_1\rangle$	R_1	R_2	
		$ \phi_2\rangle$		$ \phi_3\rangle$	
	1	R_4	R_5	$ \phi_4\rangle$	
		$ \phi_7\rangle$	$ \phi_9\rangle$		
	2	$ \phi_8\rangle$	$ \phi_5\rangle$	R_3	
			$ \phi_6\rangle$		

$$|i \pm j\rangle \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}}(|i\rangle \pm |j\rangle), \\ 0 \leq i < j \leq 2$$

$$\begin{aligned} |\phi_{1,2}\rangle &= |0\rangle|0 \pm 1\rangle \\ |\phi_{3,4}\rangle &= |0 \pm 1\rangle|2\rangle \\ |\phi_{5,6}\rangle &= |2\rangle|1 \pm 2\rangle \\ |\phi_{7,8}\rangle &= |1 \pm 2\rangle|0\rangle \\ |\phi_9\rangle &= |1\rangle|1\rangle \end{aligned}$$

FIG. 1: The basis \mathcal{B}_9 for $\mathbb{C}^3 \otimes \mathbb{C}^3$ and its rectangular representation ($\mathcal{R}_9, \{|0\rangle, |1\rangle, |2\rangle\}, \{|0\rangle, |1\rangle, |2\rangle\}, U, V$), where $\mathcal{R}_0 = \{R_i : 1 \leq i \leq 5\}$, $V_{R_1}, U_{R_2}, V_{R_3}$, and U_{R_4} are Hadamard and the other unitaries are Identities.

two classes.

Definition 1 ([4]). An *unextendable product basis (UPB)* is an OPS that is not a proper subset of any other OPS.

Note that a UPB is not necessarily a basis for the underlying product space. If \mathcal{E} is an OPS in a product space $\mathcal{H}_A \otimes \mathcal{H}_B$, denote by $\mathcal{E}_A = \{|\alpha\rangle \in \mathcal{H}_A : \exists |\beta\rangle \in \mathcal{H}_B, |\alpha\rangle|\beta\rangle \in \mathcal{E}\}$, and $\mathcal{E}_B = \{|\beta\rangle \in \mathcal{H}_B : \exists |\alpha\rangle \in \mathcal{H}_A, |\alpha\rangle|\beta\rangle \in \mathcal{E}\}$.

Definition 2 ([9]). An OPS \mathcal{E} is *irreducible* if neither \mathcal{E}_A nor \mathcal{E}_B can be partitioned into two nonempty orthogonal subsets.

Theorem 3 ([4, 5, 9]). The following OPSs are LOCC-indistinguishable:

(1) An irreducible OPB ([9]).

(2) A UPB ([4, 5]).

In fact, Ref. [9] characterizes all LOCC-indistinguishable OPBs.

Theorem 4 ([9]). An OPB cannot be reliably distinguished by LOCC if and only if it contains an irreducible

subset that spans a product space. In particular, an OPB in $3 \otimes 3$ space is LOCC-indistinguishable if and only if it is irreducible.

A main objective of this line of research is to identify additional LOCC-indistinguishable OPSs. To this end, we generalize \mathcal{B}_8 to a broader class of LOCC-indistinguishable OPSs having a similar structure. A satisfactory understanding of LOCC indistinguishability is a complete and computationally verifiable characterization of all such OPSs. Clearly, any OPS in a $1 \otimes n$ system, $n \geq 1$, is LOCC-distinguishable. It is also known [3] that the same is true for any $2 \otimes n$ system, $n \geq 1$. Thus $3 \otimes 3$ is the smallest dimension where such a characterization is not known. The main result of this Letter resolves this problem. We show that when restricted to the $3 \otimes 3$ space, the generalizations of \mathcal{B}_8 , together with irreducible OPBs and UPBs, are the only possible LOCC-indistinguishable OPSs. A key step in the proof of our characterization is to show that all irreducible OPBs in the $3 \otimes 3$ space must have a representation by rectangles similar to that of \mathcal{B}_9 .

We introduce some notions for the rest of the paper. By a slight abuse of notation, for two vectors $|\alpha\rangle$ and $|\beta\rangle$, we write $|\alpha\rangle = |\beta\rangle$ if there exists a non-zero $c \in \mathbb{C}$ such that $|\alpha\rangle = c|\beta\rangle$.

Definition 5. Two product states $|\alpha\rangle|\beta\rangle$ and $|\alpha'\rangle|\beta'\rangle$ are said to *align on the left (right)* if $|\alpha\rangle = |\alpha'\rangle$ ($|\beta\rangle = |\beta'\rangle$).

Let $m, n \geq 1$ be integers. If \mathcal{E} is an OPS in the $m \otimes n$ dimensional space and $|\mathcal{E}| = mn - 1$, then \mathcal{E} can be extended to an OPB [5]. Denote by \mathcal{E}^\perp the unique product state that extends \mathcal{E} to a basis.

Theorem 6. Let $m, n \geq 1$ be integers. An OPS described below is LOCC-indistinguishable.

- (3) An irreducible OPS \mathcal{E} in $\mathbb{C}^m \otimes \mathbb{C}^n$ with $|\mathcal{E}| = mn - 1$ such that \mathcal{E}^\perp does not align on either side with any element in \mathcal{E} .

Proof. Denote by \mathcal{H}_A and \mathcal{H}_B the state space of Alice and Bob, respectively. Suppose $\mathcal{E} = \{|\alpha_i\rangle|\beta_i\rangle : 1 \leq i \leq mn - 1\}$ and $\mathcal{E}^\perp = |\alpha_0\rangle|\beta_0\rangle$. Suppose that \mathcal{E} can be reliably distinguished by an LOCC protocol. Fix such a protocol \mathcal{P} that takes the smallest number of rounds of communication. Without loss of generality, assume that Alice sends the first message, which is the measurement outcome k of a Positive-Operator-Valued Measurement (POVM) $\mathcal{M} \stackrel{\text{def}}{=} \{M_k : \mathcal{H}_A \rightarrow \mathcal{H}'_A\}_k$, where \mathcal{H}'_A is Alice's state space after applying \mathcal{M} and the operators M_k satisfy $\sum_k M_k^\dagger M_k = I_{\mathcal{H}_A}$. If for each k , there exists $\mu_k > 0$ such that $M_k^\dagger M_k = \mu_k I_{\mathcal{H}_A}$, then $\sum_k \mu_k = 1$ and each M_k is an isometric embedding. Thus \mathcal{M} can be implemented by having Bob send the message instead: he generates a random number k with probability μ_k , sends it to Alice, who applies M_k to \mathcal{H}_A . This contradicts the assumption

that \mathcal{P} takes the smallest number of rounds. Therefore, there exists a k such that $M_k^\dagger M_k$ has $k_0 \geq 2$ number of distinct eigenvalues. Fix such a k for the rest of the proof.

Since the post-measurement states must remain orthogonal so that they can be reliably distinguished by the remaining steps of \mathcal{P} , we have $\langle \alpha_i | \langle \beta_i | (M_k^\dagger M_k \otimes I_{\mathcal{H}_B}) | \alpha_j \rangle | \beta_j \rangle = 0$, for all $1 \leq i < j \leq mn - 1$. Note that $\mathcal{E}' \stackrel{\text{def}}{=} \mathcal{E} \cup \{\mathcal{E}^\perp\}$ is an OPB, thus for each i , $1 \leq i \leq mn - 1$, there exist $\lambda_i, \lambda_i^0 \in \mathbb{C}$, such that $M_k^\dagger M_k \otimes I_B |\alpha_i\rangle |\beta_i\rangle = \lambda_i |\alpha_i\rangle |\beta_i\rangle + \lambda_i^0 |\alpha_0\rangle |\beta_0\rangle$. Applying $\langle \alpha_0 |$ on both sides, we have $\langle \alpha_0 | M_k^\dagger M_k |\alpha_i\rangle |\beta_i\rangle = \lambda_i \langle \alpha_0 | \alpha_i \rangle |\beta_i\rangle + \lambda_i^0 |\beta_0\rangle$. It follows that $\lambda_i^0 = 0$, since $|\beta_i\rangle \neq |\beta_0\rangle$. Therefore, \mathcal{E}_A is a set of eigenstates of $M_k^\dagger M_k$.

If \mathcal{E}_A does not span \mathcal{H}_A , let $|\alpha\rangle \in \mathcal{H}_A$ be a state orthogonal to $\text{span}(\mathcal{E}_A)$. Let $|\beta\rangle \in \mathcal{H}_B$ be orthogonal to $|\beta_0\rangle$. Such $|\beta\rangle$ must exist since otherwise $\dim(\mathcal{H}_B) = 1$, and \mathcal{E} would be reducible. Then $|\alpha\rangle|\beta\rangle$ is orthogonal to \mathcal{E}' , a contradiction to \mathcal{E}' being a basis for $\mathcal{H}_A \otimes \mathcal{H}_B$. Therefore, \mathcal{E}_A spans \mathcal{H}_A , and is a complete spectrum of $M_k^\dagger M_k$. It follows that \mathcal{E}_A can be partitioned into k_0 number of pair-wise orthogonal subsets, each of which corresponds to a distinct eigenvalue of $M_k^\dagger M_k$. Since $k_0 \geq 2$, this contradicts the assumption that \mathcal{E} is irreducible. Therefore, \mathcal{E} is LOCC-indistinguishable. \square

As mentioned above, the $3 \otimes 3$ space is the smallest space having LOCC-indistinguishable OPSs. We also know the following useful facts.

Proposition 7 ([5]). An OPS \mathcal{E} in $\mathbb{C}^3 \otimes \mathbb{C}^3$ is LOCC-distinguishable if $|\mathcal{E}| \leq 4$.

Theorem 8 ([4, 5]). Any UPB in $\mathbb{C}^3 \otimes \mathbb{C}^3$ must have exactly 5 elements.

In what follows, we completely characterize all LOCC-indistinguishable OPSs in the $3 \otimes 3$ space.

Theorem 9 (Main Theorem). An OPS in $\mathbb{C}^3 \otimes \mathbb{C}^3$ is LOCC-indistinguishable if and only if it belongs to one of the three classes (1), (2), and (3).

Combining the above three results, an LOCC-indistinguishable OPS in the $3 \otimes 3$ space must have precisely 5, 8, or 9 elements, each of which corresponds to belong to the classes (2), (3) and (1), respectively. Whether or not an OPS is irreducible can be checked from the pairwise inner products of the state components. The same information can be used to determine if an OPS is an UPB in the $3 \otimes 3$ space [4, 5]. Therefore, whether or not an OPS belongs to (1), (2), or (3) can be determined computationally.

To prove Main Theorem, we first generalize the rectangular representation for \mathcal{B}_9 and derive some useful properties of the generalization. Let I and J be two sets. A subset $R \subseteq I \times J$ is a *rectangle* if $R = A \times B$ for some

$A \subseteq I$ and $B \subseteq J$. If $R = A \times B$, denote by $I(R) \stackrel{\text{def}}{=} A$ and $J(R) \stackrel{\text{def}}{=} B$. A *rectangular decomposition* of $I \times J$ is a partition of $I \times J$ into rectangles. Fig. 1 illustrates a rectangular decomposition for $\{0, 1, 2\} \times \{0, 1, 2\}$. We refer to this decomposition as \mathcal{R}_9 and use the labeling scheme in the Figure for its elements.

Definition 10. Let $m, n \geq 1$ be integers, $I \stackrel{\text{def}}{=} \{0, 1, \dots, n-1\}$, and $J \stackrel{\text{def}}{=} \{0, 1, \dots, m-1\}$. Let \mathcal{E} be an OPB of a product space $\mathcal{H}_A \otimes \mathcal{H}_B$ with $\dim(\mathcal{H}_A) = n$ and $\dim(\mathcal{H}_B) = m$. A *rectangular representation* of \mathcal{E} is a quintuple $(\mathcal{R}, \alpha, \beta, U, V)$ such that:

- (a) \mathcal{R} is a rectangular decomposition of $I \times J$.
- (b) $\alpha = \{|\alpha_0\rangle, |\alpha_1\rangle, \dots, |\alpha_{n-1}\rangle\}$ is an orthonormal basis for \mathcal{H}_A , and similarly, $\beta = \{|\beta_0\rangle, |\beta_1\rangle, \dots, |\beta_{m-1}\rangle\}$ is an orthonormal basis for \mathcal{H}_B .
- (c) U assigns each $R \in \mathcal{R}$ a unitary operator U_R on $\text{span}\{|\alpha_i\rangle : i \in I(R)\}$, and similarly, V_R is a unitary operator on $\text{span}\{|\beta_j\rangle : j \in J(R)\}$.
- (d) $\mathcal{E} = \{(U_R|\alpha_i)\rangle \otimes (V_R|\beta_j)\rangle : R \in \mathcal{R}, (i, j) \in R\}$.

It can be verified by direct inspection from Fig. 1 that \mathcal{B}_9 has a rectangular representation of which the rectangular decomposition is \mathcal{R}_9 and the unitary transformations are either Identity operators or Hadamard. Removing any state other than $|1\rangle|1\rangle$ from \mathcal{B}_9 results in an LOCC-distinguishable set. The same is true for any OPB having a rectangular representation using \mathcal{R}_9 .

Proposition 11. Let \mathcal{E} be an OPB in the $3 \otimes 3$ space having a rectangular representation $(\mathcal{R}_9, \alpha, \beta, U, V)$. Suppose $|\alpha_1\rangle|\beta_1\rangle \in \mathcal{B}$ is the state corresponding to the 1×1 rectangle. Then any OPS obtained from \mathcal{E} by removing some state other than $|\alpha_1\rangle|\beta_1\rangle$ is LOCC-distinguishable.

Proof. We denote the states in \mathcal{E} by $\{|\phi_i\rangle : 1 \leq i \leq 9\}$ using the labeling scheme in Fig. 1. Without loss of generality, assume that $|\phi_1\rangle$ is the only state in \mathcal{E} missing in \mathcal{E}' . By direct inspection, the following LOCC protocol identifies an unknown input state from \mathcal{E}' . Bob starts the protocol by measuring $\{|\beta_0\rangle\langle\beta_0|, I - |\beta_0\rangle\langle\beta_0|\}$. If the measurement outcome corresponds to the first operator, Alice measures $\{|\alpha_0\rangle\langle\alpha_0|, U_{R_4}|\alpha_1\rangle\langle\alpha_1|U_{R_4}^\dagger, U_{R_4}|\alpha_2\rangle\langle\alpha_2|U_{R_4}^\dagger\}$, concluding that the input state is $|\phi_2\rangle, |\phi_7\rangle$, or $|\phi_8\rangle$ accordingly. In the other case, the protocol continues using a similar strategy. \square

We now present our Main Lemma, which characterizes irreducible OPBs (thus LOCC-indistinguishable OPBs) in terms of rectangular representations.

Lemma 12 (Main Lemma). Any irreducible OPB in the $3 \otimes 3$ space has a rectangular representation using \mathcal{R}_9 .

Proof. Let $\mathcal{E} = \{|\alpha_i\rangle|\beta_i\rangle : 1 \leq i \leq 9\}$ be an irreducible OPB in the $3 \otimes 3$ space $\mathcal{H}_A \otimes \mathcal{H}_B$. If $|\alpha_i\rangle = |\alpha_j\rangle$, denote the state by $|\alpha_{i,j}\rangle$. We will construct a rectangular representation $P = (\mathcal{R}_9, \{|0\rangle_A, |1\rangle_A, |2\rangle_A\}, \{|0\rangle_B, |1\rangle_B, |2\rangle_B\}, U, V)$ for \mathcal{E} .

We first note that there exist two states $|\alpha_1\rangle|\beta_1\rangle$ and $|\alpha_2\rangle|\beta_2\rangle \in \mathcal{E}$ that are aligned in at least one side. (In fact, we can prove that in the $3 \otimes 3$ space, there are at most 5 orthogonal product states such that no pair of them align on either side.) Assume that $|\alpha_1\rangle = |\alpha_2\rangle = |\alpha_{1,2}\rangle$; the other case would lead to the same conclusion. Then $|\beta_1\rangle \perp |\beta_2\rangle$. If there are 6 states whose component in \mathcal{H}_A is orthogonal to $|\alpha_{1,2}\rangle$, then they must span $(\text{span}\{|\alpha_{1,2}\rangle\})^\perp \otimes \mathcal{H}_B$, contradicting the assumption that \mathcal{E} is irreducible. Thus there are $|\alpha_3\rangle, |\alpha_4\rangle \in \mathcal{E}_A$ with $\langle\alpha_{1,2}|\alpha_3\rangle \neq 0$ and $\langle\alpha_{1,2}|\alpha_4\rangle \neq 0$. This implies $|\beta_3\rangle = |\beta_4\rangle$.

Repeating the above argument, we find in \mathcal{E} pairs of states $\{|\alpha_{5,6}\rangle|\beta_5\rangle, |\alpha_{5,6}\rangle|\beta_6\rangle\}$ and $\{|\alpha_7\rangle|\beta_{7,8}\rangle, |\alpha_8\rangle|\beta_{7,8}\rangle\}$. By direct inspection, $|\alpha_i\rangle|\beta_i\rangle, 1 \leq i \leq 8$, must be distinct. Denote the remaining state in \mathcal{E} by $|\alpha_9\rangle|\beta_9\rangle$.

Let $S_A \stackrel{\text{def}}{=} \{|\alpha_{1,2}\rangle, |\alpha_9\rangle, |\alpha_{5,6}\rangle\}$. We show that S_A is an orthonormal basis for \mathcal{H}_A . If $|\beta_9\rangle = |\beta_{3,4}\rangle, \{|\alpha_3\rangle|\beta_{3,4}\rangle, |\alpha_4\rangle|\beta_{3,4}\rangle, |\alpha_9\rangle|\beta_9\rangle\}$ would span $\mathcal{H}_A \otimes \text{span}\{|\beta_{3,4}\rangle\}$, contradicting \mathcal{E} being irreducible. Thus $|\beta_9\rangle \neq |\beta_{3,4}\rangle$, implying that for some $i \in \{1, 2\}, \langle\beta_i|\beta_9\rangle \neq 0$. Thus $|\alpha_9\rangle \perp |\alpha_{1,2}\rangle$. Similarly, $|\alpha_9\rangle \perp |\alpha_{5,6}\rangle$. If $|\alpha_{1,2}\rangle \not\perp |\alpha_{5,6}\rangle, \{|\beta_i\rangle : i = 1, 2, 5, 6\}$ would be mutually orthogonal, contradicting $\dim(\mathcal{H}_B) = 3$. Thus $|\alpha_{1,2}\rangle \perp |\alpha_{5,6}\rangle$. Therefore, S_A is an orthonormal basis for \mathcal{H}_A . Similarly, $S_B \stackrel{\text{def}}{=} \{|\beta_{7,8}\rangle, |\beta_9\rangle, |\beta_{3,4}\rangle\}$ is orthonormal in \mathcal{H}_B . Relabel S_A as $\{|i\rangle_A : 0 \leq i \leq 2\}$ and S_B as $\{|j\rangle_B : 0 \leq j \leq 2\}$ such that $|0\rangle_A = |\alpha_{1,2}\rangle, |0\rangle_B = |\beta_{7,8}\rangle$, etc.

Define the following unitaries as the Identity operator on the corresponding dimension 1 space: $U_{R_1}, V_{R_2}, U_{R_3}, V_{R_4}, U_{R_5}$, and V_{R_5} . Define $V_{R_1} \stackrel{\text{def}}{=} |\beta_1\rangle\langle 0| + |\beta_2\rangle\langle 1|$, $U_{R_2} \stackrel{\text{def}}{=} |\alpha_3\rangle\langle 0| + |\alpha_4\rangle\langle 1|$, $V_{R_3} \stackrel{\text{def}}{=} |\beta_5\rangle\langle 1| + |\beta_6\rangle\langle 2|$, and $U_{R_4} \stackrel{\text{def}}{=} |\alpha_7\rangle\langle 1| + |\alpha_8\rangle\langle 2|$. This completes the construction of P . By direct inspection, P is a rectangular representation of \mathcal{E} . \square

We are now ready to prove Main Theorem.

Proof of Theorem 9. Since the “if” direction is precisely the combination of Theorems 3 and 6, we need only to prove the “only if” direction. Suppose there exists an LOCC-indistinguishable OPS \mathcal{E} in the $3 \otimes 3$ space not belonging to any of (1), (2), and (3). Then by Proposition 7, Theorems 4 and 8, $5 \leq |\mathcal{E}| \leq 8$ and \mathcal{E} is extensible to an OPB \mathcal{E}' . Since \mathcal{E}' must be LOCC-indistinguishable (and thus irreducible), it has a rectangular representation using \mathcal{R}_9 , by Lemma 12. Since \mathcal{E} does not belong to Class (3), there exists a state $|\alpha\rangle|\beta\rangle$ in $\mathcal{E}' - \mathcal{E}$ not contained in the rectangle R_5 . Thus $\mathcal{E}' - \{|\alpha\rangle|\beta\rangle\}$ is LOCC-distinguishable, by Proposition 11. So must be \mathcal{E} , which

is a contradiction. Thus any LOCC-indistinguishable OPS must belong to (1), (2), or (3). \square

Our method can also be used to give an alternative proof for the fact that there is no LOCC-indistinguishable OPSs in $2 \otimes n$ spaces observed in Ref. [3]. It remains an open problem to extend our result to the complete collection of LOCC-indistinguishable OPSs in spaces of a dimension higher than $3 \otimes 3$. To this end, it may be difficult to extend our technique as the rectangular representation lemma is not true for all dimensions. For example, for any θ , $0 < \theta < \pi/2$ and $\theta \neq \pi/4$, one can show that the following OPB in the $2 \otimes 4$ dimensional space does not have a rectangular representation:

$$\begin{aligned} |\psi_{1,2}\rangle &= |0\rangle \otimes |0 \pm 1\rangle, \\ |\psi_{3,4}\rangle &= |1\rangle \otimes (\cos \theta |0\rangle \pm \sin \theta |1\rangle), \\ |\psi_{5,6}\rangle &= |0 + 1\rangle \otimes |2 \pm 3\rangle, \\ |\psi_{7,8}\rangle &= |0 - 1\rangle \otimes (\cos \theta |2\rangle \pm \sin \theta |3\rangle). \end{aligned}$$

One may generalize the notion of rectangular representations through a recursive definition. Unfortunately, there also exist OPBs that do not admit such a generalized rectangular representation. We note that an even more general concept is that of *unwindability*, defined by DiVincenzo and Terhal [16]. Therefore, a deeper understanding of unwindable OPSs may lead to a better understanding of LOCC-indistinguishable OPSs in higher dimensions.

Our result can be interpreted as an indication that LOCC protocols are quite powerful. Along this line, Walgate *et al.* [11] proved that LOCC is sufficient to reliably distinguish *two* multi-partite orthogonal pure states, even when they are entangled. When the two states are not orthogonal, LOCC protocols can reach the global optimality in either conclusive discrimination [14] or inconclusive but unambiguous discrimination [15]. Therefore, perhaps the whole class of LOCC-indistinguishable OPSs has much simpler structure than one may fear.

There are bipartite operators other than those distinguishing OPSs that do not create entanglement. Thus it remains an open problem to characterize all such operators that cannot be realized by LOCC, even in the $3 \otimes 3$ dimension case.

We observe that if an OPB has a rectangular representation $(\mathcal{R}, \alpha, \beta, U, V)$, then there is a simple LOCC protocol to identify an unknown state given *two* copies of it: the first copy is projected to the bases α and β so that the rectangle R containing the state is identified, then the second copy is measured in the product basis $\{U_R|\alpha_i\rangle \otimes V_R|\beta_j\rangle : (i, j) \in R\}$. Given an OPS, determining the number of copies of an unknown state necessary to admit an LOCC distinguishing protocol is an interesting generalization of determining if it is LOCC-distinguishable.

Another interesting generalization is to determine the optimal probability of identifying an unknown state from

a given OPS by LOCC. Finally, it remains possible that an operator cannot be realized by LOCC yet may be approximated to an arbitrary precision. Identifying such an operator or proving that none exists is a fascinating open problem.

We thank Runyao Duan and Zhengwei Zhou for discussions, and for pointing out related works. Y. Shi thanks Peter Shor for hosing him at MIT, where part of this work was done. This work was partially supported by National Science Foundation of the United States under Awards 0347078 and 0622033. Y. Feng was also partly supported by the FANEDD under Grant No. 200755, the 863 Project under Grant No. 2006AA01Z102, and the Natural Science Foundation of China under Grant Nos. 60621062 and 60503001.

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